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# Discrete equations on planar graphs

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**Abstract**
The analogue of the notion of the zero-curvature representation is given for equations of the discrete Toda lattice type on an arbitrary planar graph. Several examples are presented which generalize known integrable equations on $\mathbb{Z}^2$.

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## 1. The trivial monodromy representation

Let us remind the reader that a graph $G$ is called planar if its vertices and edges lie on a plane and the edges may cross only in the vertices; the notion of the faces is defined for such a graph. The sets of vertices, edges and faces will be denoted as $V_G$, $E_G$ and $F_G$ respectively. As a rule, we consider infinite graphs, however we always assume that each vertex and each face are incident to a finite subset of edges.

The field variables $q_v$ correspond to the vertices $v$ and are related by equations of the form

$$\Phi_v(q_v, \{q_{v'} | \, (v, v') \in E_G\}) = 0 \qquad v \in V_G. \tag{1}$$

Equations of such type can arise in the problem of the equilibrium of the particles interacting with neighbours, but other interpretations are also possible. For example, the case of a square lattice $\mathbb{Z}^2$ corresponds to the class of equations of the discrete Toda lattice type [1]; in this case one direction can be interpreted as a discrete spatial variable and the other one as discrete time. One can also consider the graphs embedded into two-dimensional surfaces, for example, the equations on cylindrical graphs can be interpreted as problems with quasi-periodic boundary conditions.

The analogue of the zero-curvature representation for equations (1) can be introduced as follows. For each face $f$ we put into the correspondence the $\psi$-function $\psi_f(\lambda)$. If the faces $f$ and $\tilde{f}$ possess a common edge $(v, v')$, then the transition from $\psi_f$ to $\psi_{\tilde{f}}$ through this edge will be defined by the formula

$$\psi_{\tilde{f}} = L_{\tilde{f}f}(v, v', \lambda)\psi_f$$

where $L$ is some operator which depends on the field variables $q_v$, $q_{v'}$ and the spectral parameter $\lambda$. The orientation of the edge is unessential, that is $L_{\tilde{f}f}(v, v') \stackrel{\text{def}}{=} L_{\tilde{f}f}(v', v)$.

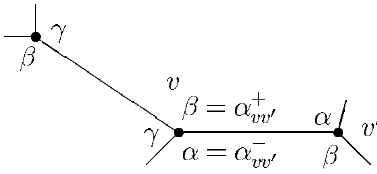


**Figure 1.** The distribution of the corner parameters.

Formally, we do not except degenerate cases when two faces possess several common edges, or when some face possesses an internal edge.

**Definition 1.** *The operators $L$ define the trivial monodromy representation for equation (1) if it is equivalent to the following condition: for each closed route through the edges of the graph the product of the corresponding operators is a scalar operator.*

An equivalent formulation can be given in terms of the geometrically dual graph $G^*$ which is defined as follows: each edge of $G$ is replaced by a new edge which connects the selected points in the adjacent faces. If the graph $G$ is connected then $G^{**} = G$. Obviously, one may assume that $\psi$-functions are associated with the vertices of $G^*$, so that definition 1 deals with the routes on the edges of the dual graph.

The necessary and sufficient condition for the monodromy to be trivial is that it must be trivial for two types of elementary routes: the transition through any edge and back again; the cyclic route around any vertex through the incidental edges. These conditions can be written as follows:

$$L_{f\tilde{f}}(v, v')L_{\tilde{f}f}(v, v') = cI \tag{2}$$

$$L_{f_N f_{N-1}}(v, v_N) \dots L_{f_2 f_1}(v, v_2)L_{f_1 f_N}(v, v_1) = cI \tag{3}$$

where $f_n$, $n \in \mathbb{Z}_N$ are the faces incidental to the given vertex $v$ and the faces $f_n$ and $f_{n-1}$ possess the common edge $(v, v_n)$; the number $N$ is called the valency of the vertex $v$ (see figure 3). It is easy to see that the starting point and direction are unessential by virtue of the first property.

## 2. Universal models

In general, the interaction of the neighbouring particles can be different for the different edges (examples of the discrete (relativistic) Toda lattices can be found in [2–9]) and therefore operators $L$ may depend on the field variables quite differently. Moreover, we cannot expect that the operators found for a given equation on a given graph will also be suitable for the other graphs. However, it turns out that some universal equations exist, for which all operators are of the same structure (up to parameters) independently on the graph. We consider several examples of the general form

$$\sum_{v'} \Phi(\alpha^+_{vv'} - \alpha^-_{vv'}, q_v - q_{v'}) = 0 \qquad (v, v') \in E_G \quad v \in V_G \tag{4}$$

where parameters $\alpha^+_{vv'}$ and $\alpha^-_{vv'}$ are assigned, in clockwise order, to the adjacent corners with the vertex $v$ and common edge $(v, v')$. We assume that these parameters are not fully arbitrary, but satisfy the condition (see figure 1)

$$\alpha^\pm_{vv'} = \alpha^\pm_{v'v}. \tag{5}$$

We will prove that equation (4) admits the trivial monodromy representation for the following instances of the function $\Phi(a, x)$:

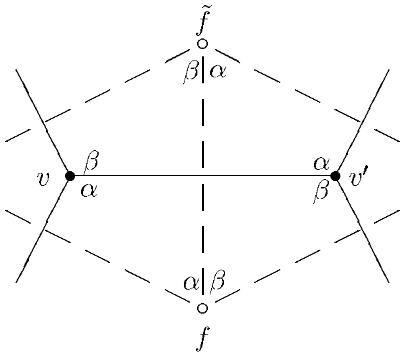


**Figure 2.** Corner parameters of the dual graph.

$$\Phi = \frac{a}{x} \tag{a}$$

$$\Phi = a \coth x \tag{b}$$

$$\Phi = \log \frac{x+a}{x-a} \tag{c}$$

$$\Phi = \log \frac{\mathrm{e}^{x+a}-1}{\mathrm{e}^{x}-\mathrm{e}^{a}}. \tag{d}$$

These functions are odd on both arguments:

$$\Phi(-a, x) = \Phi(a, -x) = -\Phi(a, x)$$

and this property allows one to interpret equation (4) as the Euler–Lagrange equation $\delta\mathcal{L} = 0$ for the functional of the form

$$\mathcal{L} = \sum_{(v,v')\in E_G} \phi(\alpha^+_{vv'} - \alpha^-_{vv'}, q_v - q_{v'}) \qquad \partial_x\phi(a, x) = \Phi(a, x).$$

It should be noted that case (b) is related to (a) by the point transformation $q \mapsto \mathrm{e}^{2q}$ and we may not consider it separately.

Cases (b) and (c) are also related, but in a more complicated way. Remarkably, the rule (5) becomes consistent with the notion of the geometrically dual graph if we assign

$$\alpha^{\mp}_{f\tilde f} = \alpha^{\pm}_{vv'}$$

where $(f, \tilde f)$ is the edge of the dual graph which is crossed with the edge $(v, v')$ (see figure 2; we denote the vertices of $G^*$ as faces of $G$, in order to save the letters). The natural question arises, are equations (4) on the dual graphs related to each other? It turns out that the relation is rather simple: it is given by the formula

$$\Phi(\alpha^+_{vv'} - \alpha^-_{vv'}, q_v - q_{v'}) = \varepsilon_{vv'f\tilde f}(q_{\tilde f} - q_f) \tag{6}$$

where the sign $\varepsilon$ depends on the mutual orientation of the edges $(v, v')$ and $(f, \tilde f)$:

$$\varepsilon_{vv'\tilde f f} = \varepsilon_{v'vf\tilde f} = -\varepsilon_{vv'f\tilde f}. \tag{7}$$

For definiteness we assume that $\varepsilon_{vv'f\tilde f} = 1$ for the configuration shown in figure 2.

Indeed, let us consider the edges and faces which are incidental to the given vertex $v$ and are enumerated as in figure 3. Formula (6) takes the form

$$\Phi(\alpha_n - \alpha_{n-1}, q_v - q_{v_n}) = q_{f_n} - q_{f_{n-1}}$$

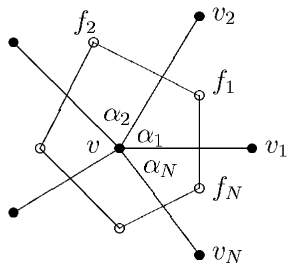


**Figure 3.** The route around the vertex.

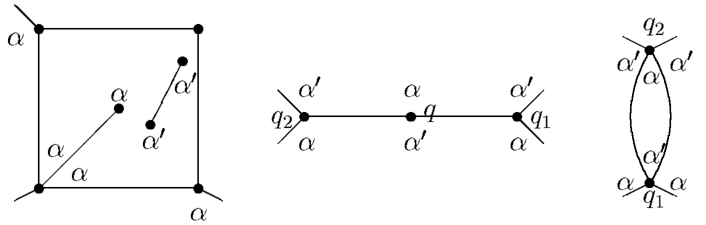


**Figure 4.** The degenerate configurations.

and summing up yields exactly equation (4). In order to obtain the dual equation we have to use the equivalent formula

$$\Phi^{-1}(\alpha^{+}_{f\tilde{f}} - \alpha^{-}_{f\tilde{f}}, q_f - q_{\tilde{f}}) = \varepsilon_{vv'f\tilde{f}}(q_v - q_{v'})$$

where $\Phi^{-1}(a, \Phi(a, x)) \equiv x$.

It is easy to check that $\Phi^{-1} = \Phi$ in cases (a) and (d) and therefore the transformation (6) maps the corresponding equations into the same type of equations on the geometrically dual graph. In case (b) one obtains $\Phi^{-1} = \frac{1}{2}\log\frac{x+a}{x-a}$ which corresponds to case (c).

We conclude this section with an analysis of several 'degenerate' subgraphs (figure 4) which can be eliminated from further consideration.

First, it is easy to see that an internal edge (such that transition through it leads to the same face) is not involved in the equation at all (since $\Phi(0, x) = 0$), so it can just be erased.

Next, if a vertex $v$ has only two neighbours $v_1$ and $v_2$, then the equation in this vertex reads

$$\Phi(\alpha - \alpha', q - q_1) + \Phi(\alpha' - \alpha, q - q_2) = 0$$

and therefore $q_1 = q_2$, while the value of $q$ remains undetermined. In such a case one can remove the vertex $v$ and glue the vertices $v_1$ and $v_2$. The equation in this new vertex is obtained by eliminating $q$ from two old equations in $v_1$ and $v_2$.

An analogous ('dual') situation takes place if the vertices $v_1$ and $v_2$ are connected by double edges. Participation of these edges in the equation is given by two cancelling terms:

$$\cdots + \Phi(\alpha - \alpha', q_1 - q_2) + \Phi(\alpha' - \alpha, q_1 - q_2) + \cdots = 0$$

so that they can be erased, after which the vertices $v_1$ and $v_2$ become unconnected.

## 3. Operators of monodromy

In this section we denote for short $\alpha = \alpha^-_{vv'}$, $\beta = \alpha^+_{vv'}$ and

$$(q,p) = \begin{cases} (q_v, q_{v'}) & \text{if} \quad \varepsilon_{vv'f\tilde f} = 1 \\ (q_{v'}, q_v) & \text{if} \quad \varepsilon_{vv'f\tilde f} = -1 \end{cases}$$

(that is, $q$ corresponds to the left-hand vertex with respect to the route from $f$ to $\tilde f$, and $p$ to the right-hand one). We also use the cyclic enumeration of the corner parameters in a given vertex, assuming $\alpha_n = \alpha^+_{vv_n}$, as in figure 3. In this notation conditions (2), (3) take the form

$$L(\lambda, \beta, \alpha, q, p)L(\lambda, \beta, \alpha, p, q) = cI \tag{8}$$

$$L_N \ldots L_2 L_1 = cI \qquad L_n = L(\lambda, \alpha_n, \alpha_{n-1}, q, q_n) \qquad n \in \mathbb{Z}_N. \tag{9}$$

**Theorem 1.** *The trivial monodromy representation for equation (4), (a) is given by the matrices*

$$L = (2\lambda - \alpha - \beta)I + (\alpha - \beta)S(q,p) \qquad S(q,p) = \frac{1}{q-p}\begin{pmatrix} q+p & 2 \\ -2qp & -q-p \end{pmatrix}. \tag{10}$$

**Proof.** The properties

$$S(q,p) = -S(p,q) \qquad S(q,p)S(q,r) = I + S(q,p) - S(q,r)$$

are valid. In particular, $S^2(q,p) = I$ which implies

$$L(\lambda, \beta, \alpha, q, p)L(\lambda, \beta, \alpha, p, q) = 4(\lambda - \alpha)(\lambda - \beta)I.$$

Next, using the property $S_m S_n = I + S_m - S_n$ where $S_n = S(q, q_n)$, one can prove by induction the formula

$$\begin{aligned} L_n \ldots L_1 = 2^{n-1}(\lambda - \alpha_{n-1}) \ldots (\lambda - \alpha_1)[(2\lambda - \alpha_n - \alpha_N)I \\ -(\alpha_n - \alpha_{n-1})S_n - \cdots - (\alpha_2 - \alpha_1)S_2 - (\alpha_1 - \alpha_N)S_1] \end{aligned}$$

and therefore the condition (9) yields

$$(\alpha_N - \alpha_{N-1})S_N + \cdots + (\alpha_2 - \alpha_1)S_2 + (\alpha_1 - \alpha_N)S_1 = 0.$$

The equivalence of this matrix equation to (4), (a) can be easily proved. □

We have already mentioned that cases (a) and (b) are point equivalent, and cases (b) and (c) are related by a duality transformation (6). However, the nonlocal character of this transformation does not allow one to rewrite the operators $L$. Their structure for cases (c) and (d) is somewhat different from (10), since it is more natural to write these equations in multiplicative rather than additive form.

**Theorem 2.** *The trivial monodromy representation for equations (4), (c) and (4), (d) (after the change* $e^q \to q$, $e^\alpha \to \alpha$*) is given by the matrices*

$$L = \frac{1}{q - p + \alpha - \beta}\begin{pmatrix} (\lambda - \beta)q - (\lambda - \alpha)p & \alpha - \beta \\ (\alpha - \beta)((\lambda - \alpha)(\lambda - \beta) - qp) & (\lambda - \alpha)q - (\lambda - \beta)p \end{pmatrix}$$

$$L = \frac{1}{\alpha p - \beta q}\begin{pmatrix} \beta(\alpha^2 - \lambda)p - \alpha(\beta^2 - \lambda)q & \beta^2 - \alpha^2 \\ \lambda(\alpha^2 - \beta^2)qp & \alpha(\beta^2 - \lambda)p - \beta(\alpha^2 - \lambda)q \end{pmatrix}$$

*respectively.*

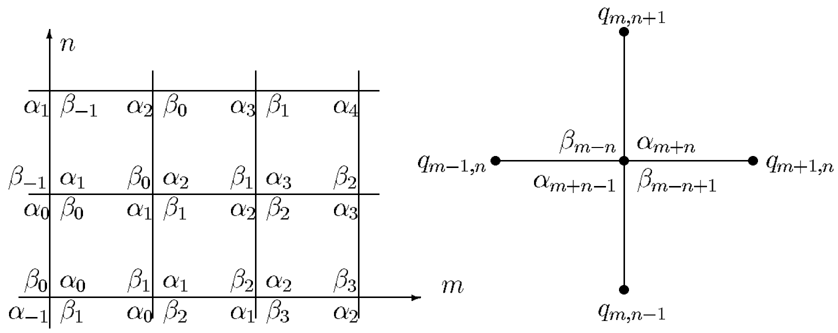


**Figure 5.** Square lattice.

**Proof.** The property (8) is proved directly, and the property (9) follows from the factorization of the form

$$L_n = A_n^{-1} M_n A_{n-1}$$

where $M_n$ is a diagonal matrix. In case (c) these matrices are

$$A_n = \begin{pmatrix} \alpha_n - \lambda + q & 1 \\ \alpha_n - \lambda - q & -1 \end{pmatrix} \qquad M_n = (\lambda - \alpha_n) \begin{pmatrix} \frac{q - q_n + \alpha_n - \alpha_{n-1}}{q - q_n - \alpha_n + \alpha_{n-1}} & 0 \\ 0 & 1 \end{pmatrix}.$$

and in case (d)

$$A_n = \begin{pmatrix} -\lambda & \alpha_n/q \\ -\alpha_n q & 1 \end{pmatrix} \qquad M_n = (\alpha_n^2 - \lambda) \begin{pmatrix} \frac{\alpha_n q_n - \alpha_{n-1} q}{\alpha_{n-1} q_n - \alpha_n q} & 0 \\ 0 & 1 \end{pmatrix}.$$

□

## 4. Examples

In this section we consider equations (4), (a) for the two simplest examples of square and hexagonal lattices. In these cases the distribution of the corner parameters can be described explicitly.

**Example 1.** Consider the square lattice with the variables $q_{m,n}$. It is easy to see that the rule (5) allows one to reconstruct the set of parameters $\alpha$ by the values given just above the axis $m$ (see figure 5).

This yields the equation

$$\frac{\beta_{m-n+1} - \alpha_{m+n-1}}{q_{m,n} - q_{m,n-1}} + \frac{\alpha_{m+n-1} - \beta_{m-n}}{q_{m,n} - q_{m-1,n}} + \frac{\beta_{m-n} - \alpha_{m+n}}{q_{m,n} - q_{m,n+1}} + \frac{\alpha_{m+n} - \beta_{m-n+1}}{q_{m,n} - q_{m+1,n}} = 0$$

which is a generalization with variable coefficients of the well known discrete Toda type lattice [7].

**Example 2.** In the case of the hexagonal lattice we will enumerate the field variables by pairs of integers, as before however, this hides the rotational symmetry. Rule (5) allows one to reconstruct all the corner parameters by the values given along the axes, each axis corresponding to the corner of one of three types (for the sake of visualization only parameters $\alpha$ are presented in figure 6; the distribution of parameters $\beta$ and $\gamma$ is obtained by rotation on $\frac{2}{3}\pi$).

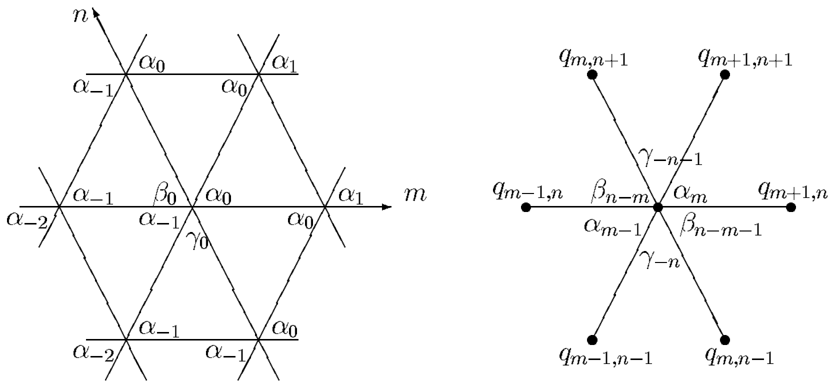


**Figure 6.** Hexagonal lattice.

The resulting discrete relativistic Toda type lattice reads

$$\frac{\beta_{n-m-1}-\gamma_{-n}}{q_{m,n}-q_{m,n-1}}+\frac{\gamma_{-n}-\alpha_{m-1}}{q_{m,n}-q_{m-1,n-1}}+\frac{\alpha_{m-1}-\beta_{n-m}}{q_{m,n}-q_{m-1,n}}$$
$$+\frac{\beta_{n-m}-\gamma_{-n-1}}{q_{m,n}-q_{m,n+1}}+\frac{\gamma_{-n-1}-\alpha_m}{q_{m,n}-q_{m+1,n+1}}+\frac{\alpha_m-\beta_{n-m-1}}{q_{m,n}-q_{m+1,n}}=0.$$

The case of constant $\alpha, \beta, \gamma$ was considered in [8, 9], and closely related equations were considered in [6].

## 5. Conclusion

We have considered discrete Toda lattice type equations on the planar graphs and demonstrated that the notion of the zero-curvature representation can be naturally formulated in terms of the geometrically dual graph. Some alternative approaches can be found, for example, in [10, 11].

Several universal examples were presented which admit such a representation for an arbitrary graph. In particular, this allows one to obtain some generalizations with variable coefficients for known equations on the square and hexagonal lattices. We have not discussed what ‘integrability’ of the equations found means. For infinite graphs with translational invariance the trace of the monodromy matrix can be used for calculation of the conservation laws, but the question relating to Liouville integrability is, of course, very delicate. For an arbitrary graph the situation is completely unclear.

Definition 1 can be adjusted for other classes of equations as well. For example, if we associate $\psi$-functions with the vertices of the graph $G$ itself and assume that the operator $L : \psi_v \to \psi_{v'}$ depends on $q_v, q_{v'}$ then the condition of the trivial monodromy leads to equations of the form

$$\Phi_f(\{q_v | \, v \in f\}) = 0 \qquad f \in F_G.$$

In the $\mathbb{Z}^2$ case this class contains the difference KdV and Liouville equations and many others which can be obtained from the nonlinear superposition principle for Bäcklund transformations, see, for example, [12]. However, at present I do not know integrable examples of this type on graphs other than[1] $\mathbb{Z}^2$.

[1] Such examples appeared in the recent papers [13, 14] which were unknown to me at the time of writing this paper.

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